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Fermat's principle and real space–time rays in absorbing media

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Abstract. Real wave packets and group velocities are considered for linear, dispersive homogeneous (but possibly anisotropic) absorptive media. In inhomogeneous media the rays are determined by the Fermat principle. Coupled with the relevant constraints this yields the Hamilton equations of geometrical optics for real rays in absorbing media.

1. Introduction

Ray methods are widely used in physics for discussion of short wave propagation in inhomogeneous media, e.g. for quantum mechanical applications (Synge 1954) and electromagnetic waves (Brandstatter 1963, Felsen and Marcuvitz 1973). The identification of rays as characteristics of the eikonal equation is well understood, and the physical import of the group velocity as describing the motion of energy packets or particles is discussed by many authors.

The problem of ray tracing in absorbing media, where the dispersion, or eikonal equations are complex is not unique, and essentially two approaches are evident in the literature. Jones (1970), and Budden and Terry (1971) generalised the Hamilton equations formally, by continuing space–time into the complex domain. This method of complex ray tracing has been widely used (see, e.g., Bennett 1974, Wang and Deschamps 1974, Connor and Felsen 1974, who give ample references to the existing literature). While mathematically consistent, in providing solutions to the eikonal equation, these methods are difficult to interpret physically. This is probably the reason for the many attempts to trace real rays in absorbing media. Various definitions for the group velocity have been suggested (Barsukow and Ginzburg 1964, Storey and Roehner 1970, Suchy 1972a, b, 1974). The latter have been claimed to give rise to non-unique results; this has been discussed by Bennett (1974), Suchy (1974), and Censor and Suchy (1976) who show that Suchy's formalism (Suchy 1972b) cannot be considered non-unique in view of Bennett's (1974) argument. Recently Censor and Suchy (1975) provided an alternative method for real ray tracing in absorbing media. All methods cited above are characterised by the fact that if absorption vanishes, the classical Hamilton equations of geometrical optics are obtained.

In the present study the determination of real rays as the path describing the motion of wave packets in absorbing media is considered by means of Fermat's principle. It is shown that the previous results (Censor and Suchy 1975) obey the Fermat principle, although the original argument was quite heuristic.

We start by discussing the primitive ideas of group propagation in homogeneous, time-independent media. The Fermat principle is discussed in space–time, and its plausibility is justified in terms of the extremal proper time associated with a wave packet which moves between two fixed points in space–time. The real ray tracing formalism proposed by Censor and Suchy (1975) is derived from Fermat's principle, coupled with the relevant constraints. These are the requirement that the path is real, and that the dispersion equation must be satisfied along the path.

2. Real group velocity and wave packets in homogeneous media

The concept of a group velocity is commonly introduced (Brillouin 1960) by the beat produced by two plane waves having slightly different $\boldsymbol{K}$:

$$\tfrac{1}{2}(\mathrm{e}^{\mathrm{i}(\boldsymbol{K}+\delta\boldsymbol{K}).\boldsymbol{X}}+\mathrm{e}^{\mathrm{i}(\boldsymbol{K}-\delta\boldsymbol{K}).\boldsymbol{X}})=\mathrm{e}^{\mathrm{i}\boldsymbol{K}.\boldsymbol{X}}\cos(\delta\boldsymbol{K}\,.\,\boldsymbol{X}), \tag{1}$$

for compactness the four-vector notation is used for space–time $\boldsymbol{X}=(\boldsymbol{x}, \mathrm{i}ct)$, and $\boldsymbol{K}=(\boldsymbol{k}, \mathrm{i}\omega/c)$, where $\boldsymbol{k}, \omega$ are real. The medium is characterised by a dispersion equation $\omega=\omega(\boldsymbol{k})$, which is represented as $F(\boldsymbol{K})=0$. In (1), a train of wave packets is described, where $\exp(\mathrm{i}\boldsymbol{K}\,.\,\boldsymbol{X})$ is the carrier, $\cos(\delta\boldsymbol{K}\,.\,\boldsymbol{X})$ is the envelope. Let a path $\boldsymbol{X}(w)$ be traced, where w is a real parameter such that $\delta\boldsymbol{K}.\boldsymbol{X}=$ constant, i.e. we follow a certain value of the envelope through space–time according to $\delta\boldsymbol{K}.\mathrm{d}\boldsymbol{X}/\mathrm{d}w=0$. Assuming small $\delta\boldsymbol{K}$,

$$F(\boldsymbol{K}\pm\delta\boldsymbol{K})=F(\boldsymbol{K})\pm\frac{\partial F}{\partial\boldsymbol{K}}\,.\,\delta\boldsymbol{K}, \qquad \frac{\partial}{\partial\boldsymbol{K}}=\left(\frac{\partial}{\partial\boldsymbol{k}}, -\mathrm{i}c\frac{\partial}{\partial\omega}\right) \tag{2}$$

and since $F=0$ for all allowed arguments, we have

$$\delta F=\frac{\partial F}{\partial\boldsymbol{K}}\,.\,\delta\boldsymbol{K}=0. \tag{3}$$

It follows that

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}=\lambda(w)\frac{\partial F}{\partial\boldsymbol{K}}, \tag{4}$$

where $\lambda(w)$ is an arbitrary, real, scalar function. Splitting (4) into space and time components, we get

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}w}=\lambda\frac{\partial F}{\partial\boldsymbol{k}}, \qquad \frac{\mathrm{d}t}{\mathrm{d}w}=-\lambda\frac{\partial F}{\partial\omega}. \tag{5}$$

Up to this point, the use of the four-vector notation was only a matter of convenience. However, we wish the velocity

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=-\frac{\partial F/\partial\boldsymbol{k}}{\partial F/\partial\omega}, \tag{6}$$

to describe the motion of physical entities, e.g. energy, or particles. The relation of space–time rays to the theory of relativity is discussed by Synge (1954). The significance of using here special relativistic ideas will be further discussed in connection with the Fermat principle, below, giving it a physical meaning for space and time varying media.

The Minkowski arc length element is defined by $\mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}=(\mathrm{d}s)^2$. Accordingly it is seen from (4) that $\mathrm{d}w\lambda(w)$ is not arbitrary, obeying the relation

$$\lambda\,\mathrm{d}w=\mathrm{d}s\left(\frac{\partial F}{\partial \boldsymbol{K}}\cdot\frac{\partial F}{\partial \boldsymbol{K}}\right)^{-1/2} \tag{7}$$

or, by defining the proper time $\mathrm{d}\tau=\mathrm{d}s/\mathrm{i}c$, the relation of $\lambda\,\mathrm{d}w$ to the proper time is established. The second line (5) is now written in the form

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=-\frac{\partial F}{\partial\omega}\mathrm{i}c\left(\frac{\partial F}{\partial \boldsymbol{K}}\cdot\frac{\partial F}{\partial \boldsymbol{K}}\right)^{-1/2} \tag{8}$$

and for real t, τ, $\partial F/\partial\omega$, c the product in parentheses in (8) must be negative. This implies $|\mathrm{d}\boldsymbol{x}/\mathrm{d}t|<c$. In physical systems where $|\mathrm{d}\boldsymbol{x}/\mathrm{d}t|\ll c$, e.g. acoustic waves, $\mathrm{d}t\cong\mathrm{d}\tau$ and the whole argument is trivial, for $|\mathrm{d}\boldsymbol{x}/\mathrm{d}t|=c$, $\mathrm{d}\tau=0$ and the argument breaks down. However, for electromagnetic waves in material media and relativistic particles the discussion is relevant.

Consider now the problem of complex $\boldsymbol{K}$, i.e. complex $\boldsymbol{k},\omega$, which arises in connection with absorbing media. Let us define

$$\begin{aligned}\boldsymbol{K}&=\mathrm{R}\boldsymbol{K}+\mathrm{i}\mathrm{I}\boldsymbol{K},\\ \mathrm{R}\boldsymbol{K}&=(\mathrm{Re}\,\boldsymbol{k},\mathrm{i}\,\mathrm{Re}\,\omega/c),\\ \mathrm{I}\boldsymbol{K}&=(\mathrm{Im}\,\boldsymbol{k},\mathrm{i}\,\mathrm{Im}\,\omega/c),\end{aligned} \tag{9}$$

where Re, Im denote the real, imaginary parts, respectively. Accordingly (1) becomes

$$\exp(-\mathrm{I}\boldsymbol{K}\cdot\boldsymbol{X}+\mathrm{i}\mathrm{R}\boldsymbol{K}\cdot\boldsymbol{X})[\cos(\mathrm{R}\delta\boldsymbol{K}\cdot\boldsymbol{X})\cosh(\mathrm{I}\delta\boldsymbol{K}\cdot\boldsymbol{X})-\mathrm{i}\sin(\mathrm{R}\delta\boldsymbol{K}\cdot\boldsymbol{X})\sinh(\mathrm{I}\delta\boldsymbol{K}\cdot\boldsymbol{X})]. \tag{10}$$

It is clear that the envelope, i.e. the terms in square brackets in (10), is constant provided $\mathrm{R}\delta\boldsymbol{K}\cdot\boldsymbol{X}$, $\mathrm{I}\delta\boldsymbol{K}\cdot\boldsymbol{X}$ are constant, or $\mathrm{R}\delta\boldsymbol{K}\cdot\mathrm{d}\boldsymbol{X}/\mathrm{d}\tau=\mathrm{I}\delta\boldsymbol{K}\cdot\mathrm{d}\boldsymbol{X}/\mathrm{d}\tau=0$, simultaneously. On the other hand, if F is analytical in its arguments K_i, $i=1,\dots,4$, (3) is valid, prescribing

$$\begin{aligned}\mathrm{R}\delta\boldsymbol{K}\cdot\mathrm{R}\frac{\partial F}{\partial\boldsymbol{K}}-\mathrm{I}\delta\boldsymbol{K}\cdot\mathrm{I}\frac{\partial F}{\partial\boldsymbol{K}}&=0,\\ \mathrm{R}\delta\boldsymbol{K}\cdot\mathrm{I}\frac{\partial F}{\partial\boldsymbol{K}}+\mathrm{I}\delta\boldsymbol{K}\cdot\mathrm{R}\frac{\partial F}{\partial\boldsymbol{K}}&=0.\end{aligned} \tag{11}$$

For $\mathrm{I}\boldsymbol{X}=0$, i.e. real $\boldsymbol{x}$, t, (11) can be satisfied for complex $\boldsymbol{K}$ if $\mathrm{I}(\partial F(\boldsymbol{K})/\partial\boldsymbol{K})=0$. The analogue of (4) is now

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}=\lambda(w)\frac{\partial F}{\partial\boldsymbol{K}},\qquad \mathrm{I}\frac{\partial F}{\partial\boldsymbol{K}}=0. \tag{12}$$

The argument (6)–(8) is still valid. Consequently a wave packet having a real group velocity can be constructed in an absorbing medium. It is noted that the carrier wave now has an exponentially decaying part, in addition to the oscillatory term.

It is possible to extend (4) into the complex domain by assuming complex space–time. This corresponds to the complex ray tracing formalism mentioned in the introduction.

Returning to real $\boldsymbol{k}$, ω in lossless media, consider arbitrary wave packets, represented as a superposition of plane waves

$$\int \mathrm{d}^4\boldsymbol{K} A(\boldsymbol{K})\delta(F)\,\mathrm{e}^{\mathrm{i}\boldsymbol{K}.\boldsymbol{X}}, \tag{13}$$

where $\delta(F)$ signifies that only values $\boldsymbol{K}$ satisfying $F(\boldsymbol{K})=0$ are admitted, $A(\boldsymbol{K})$ denotes an amplitude function. By virtue of $\delta(F)$ we actually have a three-fold integral, on a surface in $\boldsymbol{K}$ space described by $F(\boldsymbol{K})=0$. Analogously to (2), we expand F about some central value $\boldsymbol{K}_0$, keeping only the first derivative,

$$F(\boldsymbol{K})=F(\boldsymbol{K}_0)+\frac{\partial F}{\partial \boldsymbol{K}_0}\boldsymbol{.}(\boldsymbol{K}-\boldsymbol{K}_0), \tag{14}$$

and note that the last term vanishes in view of $F(\boldsymbol{K})=F(\boldsymbol{K}_0)=0$. Splitting (13) into carrier and envelope factors, we have

$$\exp(\mathrm{i}\boldsymbol{K}_0\boldsymbol{.}\boldsymbol{X})\int \mathrm{d}^4\boldsymbol{K} A(\boldsymbol{K})\delta(F)\exp[\mathrm{i}(\boldsymbol{K}-\boldsymbol{K}_0)\boldsymbol{.}\boldsymbol{X}]. \tag{15}$$

The envelope is constant subject to $(\boldsymbol{K}-\boldsymbol{K}_0)\boldsymbol{.}\boldsymbol{X}=$ constant, hence (4) is again obtained. The value $(\boldsymbol{K}-\boldsymbol{K}_0)\boldsymbol{.}\boldsymbol{X}=0$ locates the saddle points of the integral (15), thus (4) describes the motion of the region in space–time where constructive interference produces the wave packet. A similar discussion has been given by Stratton (1941, p 332) for one-dimensional waves.

For absorbing media (15) is applicable if a real group velocity is imposed. Accordingly a $\delta(\mathrm{I}(\partial F/\partial \boldsymbol{K})$ factor is included in the integral (15).

3. Real ray tracing in lossless media and Fermat's principle

The theory of geometrical optics is discussed in many books (Brandstatter 1963, Felsen and Marcuvitz 1973) hence a few introductory remarks will suffice here. Consider a system of linear partial differential equations which govern the physical problems of wave propagation in a sourceless domain

$$F_{mn}\left(\frac{\partial}{\partial \boldsymbol{X}};\boldsymbol{X}\right)f_n(\boldsymbol{X})=0, \tag{16}$$

where F_{mn} is a square matrix involving the operator

$$\frac{\partial}{\partial \boldsymbol{X}}=\left(\frac{\partial}{\partial \boldsymbol{x}},\frac{1}{\mathrm{i}c}\frac{\partial}{\partial t}\right),$$

and f_n are the field components, e.g. $\boldsymbol{E},\boldsymbol{H}$ fields for the electromagnetic case. The leading term of the asymptotic expansion used in geometrical optics is given by

$$f_n(\boldsymbol{X})=a_n(\boldsymbol{X})\,\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{X})}, \tag{17}$$

where the large parameter of this asymptotic expansion is absorbed in θ here. The amplitude a_n and the derivatives of θ are slowly varying functions of $\boldsymbol{X}$. Substituting

(17) in (16) leads to a system of homogeneous equations whose determinant must vanish, hence,

$$\det F_{mn}\left(\mathrm{i}\,\frac{\partial\theta}{\partial \boldsymbol{X}};\boldsymbol{X}\right)\equiv F\left(\frac{\partial\theta}{\partial \boldsymbol{X}};\boldsymbol{X}\right)=0. \tag{18}$$

This is the eikonal equation, and written in the form

$$F(\boldsymbol{K};\boldsymbol{X})=0, \qquad \boldsymbol{K}=\partial\theta/\partial\boldsymbol{X}=(\boldsymbol{k},\mathrm{i}\omega/c), \tag{19}$$

it is called the dispersion equation. The phase is represented in the form

$$\theta(\boldsymbol{X})=\int_{\boldsymbol{X}_0}^{\boldsymbol{X}}\boldsymbol{K}(\boldsymbol{X}')\,.\,\mathrm{d}\boldsymbol{X}', \tag{20}$$

where $\boldsymbol{X}_0$ is fixed and $\boldsymbol{X}'$ denotes the integration variable. For consistency with the definition of $\boldsymbol{K}$ in (19), the integral (20) must be independent of the choice of the path between the end points, this prescribes that the four-curl of $\boldsymbol{K}$ must vanish (Poeverlein 1962) or $\nabla\times\boldsymbol{k}=0$, $\partial\omega/\partial\boldsymbol{x}+\partial\boldsymbol{k}/\partial t=0$, equivalently. To solve (18), (19) they are usually reduced (Felsen and Marcuvitz 1973) to a system of first-order ordinary equations by writing

$$\frac{\mathrm{d}F}{\mathrm{d}w}=\sum_i\left(\frac{\partial F}{\partial K_i}\frac{\mathrm{d}K_i}{\mathrm{d}w}+\frac{\partial F}{\partial X_i}\frac{\mathrm{d}X_i}{\mathrm{d}w}\right)=0, \tag{21}$$

where summation over $i=1,\ldots,4$ denoting the components of $\boldsymbol{K},\boldsymbol{X}$, is understood, and w is a real parameter. The differential equations

$$\frac{\mathrm{d}X_i}{\mathrm{d}w}=\lambda_i(w)\frac{\partial F}{\partial K_i}, \qquad \frac{\mathrm{d}K_i}{\mathrm{d}w}=-\lambda_i(w)\frac{\partial F}{\partial X_i}, \tag{22}$$

satisfy (21) with arbitrary functions $\lambda_1(w),\lambda_2(w),\lambda_3(w),\lambda_4(w)$. It is therefore clear that *unless additional constraints are imposed,* (22) *is not unique.* On the trajectories chosen, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}w}=\frac{\partial\theta}{\partial\boldsymbol{X}}\,.\,\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}=\boldsymbol{K}\,.\,\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}, \tag{23}$$

hence

$$\theta=\int\boldsymbol{K}\,.\,\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}\,\mathrm{d}w, \tag{24}$$

as in (20). To determine those special paths, which describe the motion of the wave packet, the Fermat principle is imposed. In its simplest form (in time-independent media) the principle states that wave packets, moving between two fixed points in space, will follow the path on which the time is an extremum. Consider the variation of $\mathrm{d}\theta$ (23):

$$\delta\,\mathrm{d}\theta=\delta\left(\frac{\partial\theta}{\partial\boldsymbol{x}}\,.\,\mathrm{d}\boldsymbol{x}+\frac{\partial\theta}{\partial t}\,\mathrm{d}t\right)=\delta\left[\left(\boldsymbol{k}\,.\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}-\omega\right)\mathrm{d}t\right]. \tag{25}$$

For time-independent media ω is constant hence $\delta\,\mathrm{d}\theta=0$ implies

$$\delta\left(\boldsymbol{k}\,.\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)\mathrm{d}t+\boldsymbol{k}\,.\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\delta\,\mathrm{d}t=0. \tag{26}$$

Fermat's principle means that $\delta\, dt = 0$, i.e. the time must be an extremum. Hence $\delta(\boldsymbol{k}\,.\,d\boldsymbol{x}/dt) = 0$.

For a time-independent medium we therefore have

$$\delta \int \boldsymbol{k}\,.\,d\boldsymbol{x} = \delta \int \boldsymbol{k}\,.\,\frac{d\boldsymbol{x}}{dt}\,dt = 0, \tag{27}$$

where the variation affects $\boldsymbol{k}$, $d\boldsymbol{x}$, but not the integration variable dt. In (24), if we fix the end points, then the time at the two ends of the path is fixed, and cannot be used as the quantity that is extremised. If we adopt $\delta\, d\theta = 0$ as the general statement of Fermat's principle, then, as in (26)

$$\delta\left(\boldsymbol{K}\,.\,\frac{d\boldsymbol{X}}{dw}\right) dw + \boldsymbol{K}\,.\,\frac{d\boldsymbol{X}}{dw}\delta\, dw = 0. \tag{28}$$

However, the condition $\delta\, dw = 0$ does not have a physical interpretation. But if the relation of w to the proper time τ is recognised, (28) becomes physically meaningful. We choose to identify $w = \tau$ as the proper time, thus $d\boldsymbol{X}/d\tau$ becomes the four-velocity. From (28), the condition that $\delta\, d\tau = 0$, i.e. that the path will be such that the proper time is extremised, prescribes

$$\delta \int \boldsymbol{K}\,.\,\frac{d\boldsymbol{X}}{d\tau}\,d\tau = 0. \tag{29}$$

At this point it is observed that $F = 0$ (19) must also be satisfied, hence we have a constraint associated with (29). Using a Lagrange multiplier function and adding the constraint to the integrand (29), we get

$$\delta \int \left(\boldsymbol{K}\,.\,\frac{d\boldsymbol{X}}{d\tau} - \lambda(\tau)F(\boldsymbol{K};\boldsymbol{X})\right) d\tau = 0. \tag{30}$$

By varying $\boldsymbol{K}$, $\boldsymbol{X}$ and observing that the variation vanishes at the end points, we obtain

$$\frac{d\boldsymbol{X}}{d\tau} = \lambda(\tau)\frac{\partial F}{\partial \boldsymbol{K}}, \qquad \frac{d\boldsymbol{K}}{d\tau} = -\lambda(\tau)\frac{\partial F}{\partial \boldsymbol{X}}, \tag{31}$$

which are recognised as a special case of (22). The value of λ (31) is obtained from (7). Splitting (31) into space and time components and dividing by dt, we obtain

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= -\frac{\partial F/\partial \boldsymbol{k}}{\partial F/\partial \omega}, \qquad & \frac{d\boldsymbol{k}}{dt} &= \frac{\partial F/\partial \boldsymbol{x}}{\partial F/\partial \omega}, \\ \frac{d\omega}{dt} &= -\frac{\partial F/\partial t}{\partial F/\partial \omega}, \qquad & \frac{dt}{d\tau} &= -\lambda\frac{\partial F}{\partial \omega}. \end{aligned} \tag{32}$$

The last equation is uncoupled, and the solution of the others describes the path of the wave packet $\boldsymbol{x}(t)$, and $\boldsymbol{k}(t)$, $\omega(t)$ of the carrier as they vary along the path.

4. Real ray tracing in absorbing media

In this section the ray equations for real rays in absorbing media will be discussed. One way of dealing with complex $\boldsymbol{K}$ is to admit complex $\boldsymbol{X}$, such that $F(\boldsymbol{K}, \boldsymbol{X}) = 0$ in (18), (19) and $\theta(\boldsymbol{X})$ in (20) are analytic in the complex variables K_i, X_i, $i = 1, \ldots, 4$. The extremum (29) now corresponds to saddle points, rather than maxima or minima. Although mathematically everything is in order, the physical interpretation of Fermat's principle and the concept of a complex group velocity are not clear, except when the complex ray intersects real space–time.

However, the discussion of wave packets in homogeneous media (12), shows that real rays can be defined in absorbing media, describing the motion of wave packets with real group velocities.

It is therefore necessary to confine $\boldsymbol{X}(w)$, describing the path, so that $\mathrm{I}\boldsymbol{X}(w)=0$. This is a vector constraint, hence we add it into the Fermat principle integral with a vector Lagrange function $\mathrm{I}\boldsymbol{\Lambda}$. We may also define $\boldsymbol{\Lambda}=\mathrm{R}\boldsymbol{\Lambda}+\mathrm{I}\boldsymbol{\Lambda}$, such that $\mathrm{R}\boldsymbol{\Lambda}=0$ and add $\boldsymbol{\Lambda}\,.\,\boldsymbol{X}$ in the integral, provided $\mathrm{I}\boldsymbol{\Lambda}\,.\,\mathrm{R}\boldsymbol{X}=0$ is stipulated.

Accordingly, the Fermat principle is written in the form

$$\delta\theta=0=\delta\int\left(\boldsymbol{K}\,.\,\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}-\lambda F(\boldsymbol{K};\boldsymbol{X})-\boldsymbol{\Lambda}\,.\,\boldsymbol{X}\right)\mathrm{d}w, \tag{33}$$

which leads to the set of equations

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}=\lambda(w)\frac{\partial F}{\partial\boldsymbol{K}},\qquad \frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}w}=-\lambda(w)\frac{\partial F}{\partial\boldsymbol{X}}-\boldsymbol{\Lambda}(w). \tag{34}$$

Since $\mathrm{I}\boldsymbol{X}=0$ the end points of the integral are taken in real space–time. Multiplying the first equation (34) by $\boldsymbol{\Lambda}$ and noting that $\boldsymbol{\Lambda}\,.\,\mathrm{I}\boldsymbol{X}=0$, we obtain

$$\mathrm{R}\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}=\mathrm{R}\left(\lambda(w)\frac{\partial F}{\partial\boldsymbol{K}}\right),\qquad \mathrm{I}\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}w}=\mathrm{I}\left(\lambda(w)\frac{\partial F}{\partial\boldsymbol{K}}\right)=0. \tag{35}$$

We now identify $\mathrm{R}\,\mathrm{d}\boldsymbol{X}$ as the real space–time element, and accordingly w is identified with τ, the proper time, so that $\delta\,\mathrm{d}w=0$, (28) gives the Fermat principle its physical interpretation as discussed above.

We are free to choose $\mathrm{Im}\,\lambda=0$, hence (35) can be written as

$$\frac{\mathrm{dR}\boldsymbol{X}}{\mathrm{d}\tau}=\lambda(\tau)\mathrm{R}\frac{\partial F}{\partial\boldsymbol{K}},\qquad \mathrm{I}\frac{\partial F}{\partial\boldsymbol{K}}=0, \tag{36}$$

where (36) coincides with the condition given in (12), and as in (7),

$$\lambda=-\mathrm{i}c\left(\mathrm{R}\frac{\partial F}{\partial\boldsymbol{K}}\,.\,\mathrm{R}\frac{\partial F}{\partial\boldsymbol{K}}\right)^{-1/2},$$

implying $|\mathrm{d}\boldsymbol{x}/\mathrm{d}t|<c$.

The determination of $\boldsymbol{\Lambda}$ in (34) is facilitated by observing that $\mathrm{I}(\partial F/\partial\boldsymbol{K})$ must vanish along the path, i.e.

$$\mathrm{I}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial F}{\partial\boldsymbol{K}}=0=\mathrm{I}\left(\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}\tau}\,.\,\frac{\partial}{\partial\boldsymbol{K}}\frac{\partial F}{\partial\boldsymbol{K}}+\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau}\,.\,\frac{\partial}{\partial\boldsymbol{X}}\frac{\partial F}{\partial\boldsymbol{K}}\right). \tag{37}$$

Substituting (34) subject to $\mathrm{I}\boldsymbol{X}=0$, $\mathrm{R}\boldsymbol{\Lambda}=0$, $\mathrm{Im}\,\lambda=0$, we obtain

$$\mathrm{I}\boldsymbol{\Lambda}=-\lambda\left(\mathrm{R}\frac{\partial^2F}{\partial\boldsymbol{K}\partial\boldsymbol{K}}\right)^{-1}\,.\,\mathrm{I}\left(\frac{\partial F}{\partial\boldsymbol{X}}\,.\,\frac{\partial^2F}{\partial\boldsymbol{K}\partial\boldsymbol{K}}-\frac{\partial F}{\partial\boldsymbol{K}}\,.\,\frac{\partial^2F}{\partial\boldsymbol{X}\partial\boldsymbol{K}}\right). \tag{38}$$

Equivalently, we can express the result (38) by observing that $\mathrm{I}(\partial F/\partial \boldsymbol{K}) = 0$, (36) implies

$$\operatorname{Im}\frac{\partial F/\partial \boldsymbol{k}}{\partial F/\partial \omega} = 0, \tag{39}$$

and therefore

$$\operatorname{Im}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial F/\partial \boldsymbol{k}}{\partial F/\partial \omega}\right) = 0. \tag{40}$$

This has been done by Censor and Suchy (1975) based on somewhat physical arguments, without using the Fermat principle. It is then shown that

$$\begin{aligned}
&\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\frac{\partial F/\partial \boldsymbol{k}}{\partial F/\partial \omega}, \qquad \operatorname{Im}\boldsymbol{v} = 0,\\
&\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = \frac{\partial F/\partial \boldsymbol{x}}{\partial F/\partial \omega} + \mathrm{i}\boldsymbol{\beta},\\
&\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\partial F/\partial t}{\partial F/\partial \omega} + \mathrm{i}\boldsymbol{v}\cdot\boldsymbol{\beta},\\
&\boldsymbol{\beta} = -\left[\operatorname{Re}\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{k}} + \frac{\partial \boldsymbol{v}}{\partial \omega}\boldsymbol{v}\right)\right]^{-1}\operatorname{Im}\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{k}}\cdot\frac{\partial F/\partial \boldsymbol{x}}{\partial F/\partial \omega} - \frac{\partial \boldsymbol{v}}{\partial \omega}\frac{\partial F/\partial t}{\partial F/\partial \omega}\right).
\end{aligned} \tag{41}$$

5. A simple example

As an illustration, we consider the Gaussian plane pulse, discussed by Connor and Felsen (1974), in connection with the complex ray tracing theory. At the boundary $z = 0$

$$u(0, t) = \exp[-\mathrm{i}\omega_0 t - (t/2\alpha)^2], \qquad \alpha, \omega_0 > 0, \tag{42}$$

is given, where α, ω_0 are real constants. For $z > 0$ we have a wave packet

$$u(z, t) = \frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega \, \exp[-(\omega - \omega_0)^2 + \mathrm{i}kz - \mathrm{i}\omega t], \tag{43}$$

with the proviso that $F(k, \omega) = 0$ is satisfied for the medium at hand. Since we are dealing with a one-dimensional case (scalar k), the dispersion equation can be written as $k = k(\omega)$ (Connor and Felsen 1974). For α large enough (14) applies in the form

$$k(\omega) = k(\omega_s) + \frac{\mathrm{d}k}{\mathrm{d}\omega_s}(\omega - \omega_s), \qquad \omega_s = \omega_0 + \mathrm{i}\mu, \tag{44}$$

where μ is real, such that $\mathrm{d}k/\mathrm{d}\omega_s$ is real. Substituting (44) in (43) we obtain the analogue of (15),

$$\begin{aligned}
&u(z, t) = \exp(\mathrm{i}k_s z - \mathrm{i}\omega_s t)\frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega \, \exp[-\alpha^2(\omega - \omega_0)^2 - \mathrm{i}(\omega - \omega_s)\theta]\\
&\theta = t - \frac{\mathrm{d}k}{\mathrm{d}\omega_s}z, \qquad k_s = k(\omega_s),
\end{aligned} \tag{45}$$

displaying the carrier and the envelope, the latter being constant on $\theta = \text{constant}$.

Performing the integration we get

$$u(z, t) = \exp(\mathrm{i}k_s z - \mathrm{i}\omega_s t) \exp[-\mu\theta - (\theta/2\alpha)^2], \tag{46}$$

which reduces to (42) for $z = 0$. It is possible to rewrite (46) in the form

$$u(z, t) = \exp[\mathrm{i}k_0 z - \mathrm{i}\omega_0 t - (\theta/2\alpha)^2], \tag{47}$$

where from (44),

$$k_0 = k(\omega_0) = k_s + \mu \frac{\mathrm{d}k}{\mathrm{d}\omega_s}, \tag{48}$$

and k_0 is complex, in absorbing media. We have therefore shown how the pulse moves in an absorbing medium, with a real group velocity and a complex $\boldsymbol{K}$ for the carrier.

The model given here, as seen from (14) depends on the fact that terms containing higher derivatives can be neglected. This is true when the dispersion is small, such that higher derivatives of F are negligible, or when we have a narrow band spectrum such that powers of $(\boldsymbol{K} - \boldsymbol{K}_0)$ can be neglected. A numerical analysis performed by Terina (1972) shows the effects of dispersion and absorption when the present assumptions do not hold any more. The interesting result is that the group velocity and central (carrier) frequency vary as the wave packet moves into the medium. To see the effect of higher derivatives, the integral (43) can be approximated by the saddle point method. The saddle points are given by

$$\omega_s = \omega_0 - \frac{\mathrm{i}}{2\alpha^2}\left(t - z\frac{\mathrm{d}k}{\mathrm{d}\omega_s}\right), \tag{49}$$

and

$$\begin{aligned} &u(z, t) \sim 2^{1/2}\alpha\, \mathrm{e}^{\mathrm{i}S}(2\alpha^2 - \mathrm{i}z\, \mathrm{d}^2k/\mathrm{d}^2\omega_s)^{-1/2}, \\ &S = k_s z - \omega_s t + \mathrm{i}\alpha^2(\omega_s - \omega_0)^2. \end{aligned} \tag{50}$$

For a lossless medium, we define

$$\omega_s = \omega_0, \qquad t = z\frac{\mathrm{d}k}{\mathrm{d}\omega_s}, \tag{51}$$

then (49) is satisfied. This means that $e^{\mathrm{i}S}$ with $S = k_0 z - \omega_0 t$ is the carrier, and the amplitude changes with z according to the term in parentheses in (50), i.e. depending on $\mathrm{d}^2k/\mathrm{d}\omega_0^2$. For a given α and $k(\omega)$, the distance z for which the amplitude is approximately constant can be estimated. For absorbing media $\omega_s = \omega_0 + \mathrm{i}\mu$ is chosen so that $\mathrm{d}k/\mathrm{d}\omega_s$ is real. According to (49),

$$\mu = \frac{t - z(\mathrm{d}k/\mathrm{d}\omega_s)}{2\alpha^2} \tag{52}$$

hence a different real velocity is involved, namely $\mathrm{d}z/\mathrm{d}t = (\mathrm{d}k/\mathrm{d}\omega_s)^{-1}$, and affects the path (52), as well as the amplitude, as seen when z is substituted from (52) into the parentheses in (50). Again this gives an estimate for the cases for which the approximation (14) is valid.

6. Concluding remarks

The problem of defining a wave packet and a real group velocity in absorbing media has been considered. The feasibility has been discussed for homogeneous and time-independent media. Using the extended Fermat principle, whose physical import is discussed, real ray tracing in weakly inhomogeneous and time varying media has been considered. The end products are equations (34), (38) or equivalently (41), the latter have been obtained previously, but are here justified as solutions of the dispersion equation which also satisfy Fermat's principle. Other methods of real ray tracing have been proposed in the past, e.g. by Suchy (1972b). Like all schemes based on the solution of the dispersion equation, including the present theory, they can be considered as special cases of the general complex ray tracing formalism; however, the present method ensures that the group velocity is a real velocity, satisfying the requirement that it is properly related to a four-velocity, and that $|\mathrm{d}\boldsymbol{x}/\mathrm{d}t| < c$, and simultaneously also satisfies the Fermat principle.

The formalism is applicable for any branch of physics where wave equations are given for absorbing media. Numerical computations, especially in connection with ionospheric propagation will be considered in the future.

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